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The Variational Principle for Non-Self-Adjoint Electromagnetic Problems

CHUN HSIUNG CHEN AND CHUEN-DER LIEN

***Abstract*—A systematic and intuitive procedure is proposed to derive the variational (or stationary) principle for non-self-adjoint electromagnetic problems with various boundary conditions. Several physical interpretations of this principle in terms of generalized reactions, time-average stored energy, and reactive powers, respectively, are discussed in detail. This general variational principle which makes the generalized reactions a stationary value is actually an extension of the least action principle in physics. The applications of the principle to establish the variational expressions for a waveguide, a cavity resonator, and a lossy one-dimensional inhomogeneous slab are presented.**

I. Introduction

ALMOST ALL physical problems can be formulated mathematically in terms of differential equations, integral equations, or variational equations [1], [2]. Before the advent of computer, the applicability of the variational formulation is somewhat limited, because its solution eventually has to go back to that of solving the differential equations. But in recent years, the variational formulation has received much attention for three reasons. The primary reason is that we have computers capable of solving the variational equations directly and practically, using the direct methods [2], [3], such as Ritz method and finite-element method. The second reason is that the variational formulation itself also permits a physical interpretation, thus, it may supply another picture to a physical problem. The third reason is that the variational formulation can be used not only for computing the field but also for establishing the stationary formula of a quantity such as the eigenvalue. Of course the variational formulation is not suitable in discussing the problem for which the functional does not exist [4]. Actually, only

Manuscript received October 19, 1979; revised March 19, 1980.
The authors are with the Department of Electrical Engineering, National Taiwan University, Taipei, Taiwan, Republic of China.

0018-9480/80/0800-0878$00.75 ©1980 IEEE

when the functional exists can the variational formulation have physical and mathematical significance.

In physics, both differential formulation (Newton's law) and variational formulation (the principle of least action) have been utilized extensively in studying a mechanical system. But the variational formulation of an electromagnetic system is still not well-established in comparison with the differential formulation (Maxwell's equations) of the same problem. With a few exceptions [4]–[7], most investigations on the variational formulation are mainly concerned with the self-adjoint electromagnetic problems. The purpose of this paper is to establish a general variational principle (formulation) for dealing with a non-self-adjoint electromagnetic system and then to present a physical interpretation of this novel principle.

In the field of electromagnetism, some derivations and applications of the variational formulation have been reported recently [8]–[23]. Rumsey [8] and Harrington [9] have created the reaction concept for deriving the stationary expressions in an isotropic medium. Konrad [10] has investigated the variational expressions in an anisotropic medium, while Morishita and Kumagai [11], [12] have studied the same problem, using the principle of least action. There are many authors who have derived the variational expressions for each specific problem [13]–[23]. But all these investigations are mainly concerned with the self-adjoint problems (with respect to either real- or complex-type inner product). Although a few authors [4], [5], [24] have proposed a scalar theory for handling some specific non-self-adjoint problems, the theory is still not powerful enough in dealing with the general problems for a vector field. While the variational expressions for a non-self-adjoint vector field have been obtained by means of "transpose operator and field" [7], it is still worthwhile to have an intuitive derivation and physical insight of these expressions on the other hand. Moreover, only boundary conditions of Dirichlet and/or Neumann types are considered by the previous investigations. There is still little information concerning more general types of boundary conditions.

In this paper, a systematic and intuitive procedure is suggested to derive the variational expressions for a non-self-adjoint electromagnetic system. The generalized reaction concept is introduced to interpret the general variational principle for this electromagnetic system. The principle is then applied to the problems of establishing the variational expressions for a waveguide, a cavity resonator, and a lossy one-dimensional inhomogeneous slab.

II. Variational Formulation of a Non-Self-Adjoint Problem

The variational solution of a non-self-adjoint problem

$$Lf = s \tag{1}$$

will be summarized in this section. Although only the electromagnetic field problem is considered in this study, the theory is, in general, applicable to any non-self-adjoint problem in physics and mathematics. In (1), L is a non-self-adjoint linear operator for describing a physical or mathematical problem, f is the field (unknown function) to be determined, and s is a given (known) source function.

A method of solving the original non-self-adjoint problem (1) is to introduce an auxiliary problem, the adjoint problem [4]–[6], [24], [25] as follows:

$$L^a f^a = s^a \tag{2}$$

where L^a is the adjoint operator of L, f^a is another unknown function (the adjoint field) to be determined, and s^a is another known source function.

Both real- and complex-type inner products [25] will be considered in this study. The latter is accepted conventionally by many authors and is easily interpreted as the complex power physically, however, the former is more convenient in mathematical manipulation [26]. Either real- or complex-type inner product of two vectors u and w is defined as a scalar $\langle u, w\rangle$ such that

$$\langle u,w\rangle = \langle w,u\rangle^p$$
$$\langle \sigma_1 u_1 + \sigma_2 u_2, w\rangle = \sigma_1^p\langle u_1, w\rangle + \sigma_2^p\langle u_2, w\rangle \tag{3a}$$

where u_1, u_2 are vectors and σ_1, σ_2 are scalars. For the complex-type inner product the superscript p should be interpreted as "complex conjugate*" so that

$$\sigma_i^p = \sigma_i^*, \qquad i = 1,2$$
$$\langle u,w\rangle = \langle w,u\rangle^p = \langle w,u\rangle^* \tag{3b}$$

while for the real-type inner product p should represent "no operation" and should be removed and interpreted as

$$\sigma_i^p = \sigma_i, \qquad i = 1,2$$
$$\langle u,w\rangle = \langle w,u\rangle^p = \langle w,u\rangle. \tag{3c}$$

The symbol p will always have the above interpretation throughout this study.

We now conduct a systematic and intuitive derivation of the variational formulation for solving both original and adjoint problems, equations (1) and (2), simultaneously. The idea is to express the left-hand side of the equation

$$\langle \delta f^a, Lf - s\rangle + \langle L^a f^a - s^a, \delta f\rangle = 0 \tag{4}$$

as the first variation δI of some functional I [2], where δf and δf^a are the variations of f and f^a, respectively. Then it can be shown that the problem of solving f and f^a simultaneously from (1) and (2) is completely equivalent to that of determining the stationary functions (both f and f^a) from the following variational equation:

$$\delta I(f, f^a) = 0$$
$$I(f, f^a) = \langle f^a, Lf\rangle - \langle s^a, f\rangle - \langle f^a, s\rangle. \tag{5}$$

Note that for the problems defined by differential operators L and L^a with their boundary conditions $B(f)=0$ and $B^a(f^a)=0$ regarded as essential ones, the stationary functions f and f^a of (5) should also be subject to the constraints $B(f)=0$ and $B^a(f^a)=0$, respectively. However, if these boundary conditions are regarded as natural ones, then some modifications should be made on (4) and (5), as demonstrated in Section III, with the stationary functions f and f^a subject to no constraints on the boundary.

Note also that with the symbol $\langle , \rangle$ defined by (3), the variational formulations for real- and complex-type inner products can both be written in the same form as indicated by (5). The expression in (5) is identical to that of the stationary principle [4], [24] and that of another derivation [5]. The expression (5), of course, includes that adopted in the previous investigation of the self-adjoint problem [27].

It seems that both f (the desired field) and f^a (the adjoint or auxiliary field) have to be solved simultaneously in the variational problem (5). However, the process of determining both f and f^a can actually be decoupled, as explained later, when Rayleigh–Ritz or finite element method is employed in the solution.

Let us express the solution in the form

$$f=\sum_{n=1}^{N} C_n \phi_n$$

$$f^a=\sum_{n=1}^{N} C_n^a \phi_n^a \tag{6}$$

where ϕ_n and ϕ_n^a are known functions, and C_n and C_n^a are constants to be determined. Note that both ϕ_n and ϕ_n^a may or may not form the bases of the domains of L and L^a, respectively, however they should be linearly independent and should form the complete sets as N approaches infinity. By substituting (6) into (5) or (4) and adjusting C_n and C_n^a such that $\delta I(f,f^a)=0$, one obtains two decoupled systems as follows:

$$\sum_{n=1}^{N} \langle \phi_m^a, L\phi_n \rangle C_n = \langle \phi_m^a, s\rangle$$

$$\sum_{n=1}^{N} \langle \phi_m, L^a\phi_n^a \rangle C_n^a = \langle \phi_m, s^a\rangle, \qquad m=1,2,\cdots,N. \tag{7}$$

The positive integer N in (6) and (7) may be finite or infinite if an approximate or exact solution is to be determined. The fact that C_n and C_n^a are decoupled in (7) has greatly simplified the process of determining the stationary functions f and f^a from (5).

The discrete systems (7) from the Ritz method are identical, in form, to those from the moment (or Petrov–Galerkin's) method [26] of simultaneously solving the original and adjoint problems (1) and (2). Since there is a basic difference or mathematical distinction between these two methods [4], the above statement is meaningful only when both methods can make sense. If the functions ϕ_n and ϕ_n^a are selected from the eigenfunctions of L and L^a, respectively, then the systems (7) will reduce to the conventional ones derived from the method of eigenfunction expansion [25]. If f^a in (6) is expanded into a series of ϕ_n instead of ϕ_n^a, then the resultant systems obtained will be identical to those from the Galerkin's (or Bubnov–Galerkin's) method.

The introduction of the auxiliary problem (2) for supplementing the original problem (1) has an interesting physical interpretation as follows:

$$\langle f^a, s\rangle = \langle f^a, Lf\rangle = \langle L^a f^a, f\rangle = \langle s^a, f\rangle. \tag{8}$$

This is the generalized reciprocity theorem which states that the generalized reaction of the adjoint field f^a on the source s of the original problem is identical to that of the original field f on the source s^a of the adjoint problem. The term $\langle f^a, s\rangle$, for example, may be interpreted as a generalized reaction since it can be reduced to the conventional reaction [8] if the real-type inner product is employed.

The general reciprocity theorem (8), in the case of a self-adjoint problem using the real-type inner product, has an important result as follows. By setting $f=f_1$, $s=s_1$, $f^a=f_2$, $s^a=s_2$, one then has the conventional reciprocity theorem [28] of relating the reactions between two different problems: (f_1, s_1) and (f_2, s_2), i.e., $\langle f_2, s_1\rangle = \langle f_1, s_2\rangle$.

III. Variational Principle for Electromagnetic System

The variational principle for dealing with electromagnetic field problems will be investigated in this section. This study is mainly concerned with the solution of the non-self-adjoint electromagnetic problem (the original problem) described by the equations

$$-\nabla\times\bar{E}=j\omega\,\bar{\bar{\mu}}\cdot\bar{H}+\bar{M}$$

$$\nabla\times\bar{H}=j\omega\,\bar{\bar{\epsilon}}\cdot\bar{E}+\bar{J}, \qquad \text{in } V$$

$$\bar{B}(\bar{E},\bar{H})=0, \qquad \text{on } S \tag{9}$$

where V is a closed region bounded by the closed surface S, Fig. 1. In (9), $\bar{E}$ and $\bar{H}$ are the electric and magnetic fields to be determined, $\bar{J}$ and $\bar{M}$ are the known source distributions (electric and magnetic current densities) within V, $\bar{\bar{\epsilon}}$ and $\bar{\bar{\mu}}$ are the permittivity and permeability tensors (dyadics) of the medium contained in V, and $\bar{B}(\bar{E},\bar{H})=0$ is the required boundary condition on the surface S. The anisotropic medium in V may be lossy and inhomogeneous, thus, the tensors $\bar{\bar{\epsilon}}$ and $\bar{\bar{\mu}}$ are in general complex quantities and functions of the position. Only time-harmonic variation of the form $e^{j\omega t}$ will be discussed in this study.

As before, an auxiliary adjoint problem

$$-\nabla\times\bar{E}^a=j\omega\,\bar{\bar{\mu}}^a\cdot\bar{H}^a+\bar{M}^a$$

$$\nabla\times\bar{H}^a=j\omega\,\bar{\bar{\epsilon}}^a\cdot\bar{E}^a+\bar{J}^a, \qquad \text{in } V$$

$$\bar{B}^a(\bar{E}^a,\bar{H}^a)=0, \qquad \text{on } S \tag{10}$$

is introduced to supplement the original problem (9). In (10), $\bar{E}^a$ and $\bar{H}^a$ are the unknown adjoint fields, $\bar{J}^a$ and $\bar{M}^a$ are known source distributions, and $\bar{B}^a(\bar{E}^a,\bar{H}^a)=0$ is the adjoint boundary condition. Note that the condition $\bar{B}^a(\bar{E}^a,\bar{H}^a)=0$ in (10) should be chosen according to $\bar{B}(\bar{E},\bar{H})=0$ in (9), and the dyadics $\bar{\bar{\epsilon}}^a$ and $\bar{\bar{\mu}}^a$ should be equal to $(\bar{\bar{\epsilon}}^T)^p$ and $(\bar{\bar{\mu}}^T)^p$, respectively, in order to make (10) the adjoint of (9). The symbol T denotes "transpose" and p again denotes no operation or complex conjugate* as discussed in (3).

The inner product, for studying an electromagnetic system, of vectors $\bar{U}$ and $\bar{W}$ is defined as

$$\langle \bar{U}, \bar{W}\rangle = \int_V \bar{U}^p\cdot\bar{W}\,dV. \tag{11}$$

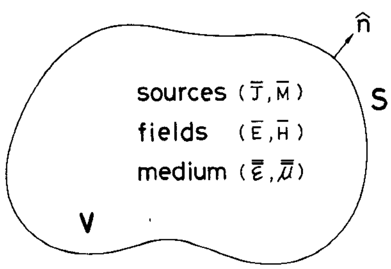

Fig. 1. Geometry of non-self-adjoint electromagnetic problem (original problem).

Two kinds of boundary conditions are discussed, i.e.,

$$\bar{B}_1(\bar{E})=\hat{n}\times\bar{E}-\bar{\gamma}=0$$
$$\bar{B}_1^a(\bar{E}^a)=\hat{n}\times\bar{E}^a-\bar{\gamma}^a=0, \qquad \text{on } S \tag{12}$$

and

$$\bar{B}_2(\bar{E},\bar{H})=\bar{H}\times\hat{n}-j\omega\,\bar{\bar{\alpha}}\cdot\bar{E}-\bar{\beta}=0$$
$$\bar{B}_2^a(\bar{E}^a,\bar{H}^a)=\bar{H}^a\times\hat{n}-j\omega\,\bar{\bar{\alpha}}^a\cdot\bar{E}^a-\bar{\beta}^a=0, \qquad \text{on } S. \tag{13}$$

In (12) and (13), $\bar{B}_i=0$ and $\bar{B}_i^a=0$, $i=1$, 2, are the boundary conditions associated with the original and adjoint problems (9) and (10), respectively, $\hat{n}$ is the unit outward normal vector to the surface S (Fig. 1), $\bar{\gamma}$, $\bar{\gamma}^a$, $\bar{\beta}$, $\bar{\beta}^a$ are given constant vectors, and $\bar{\bar{\alpha}}$, $\bar{\bar{\alpha}}^a$ are given constant dyadics. To make the problem (10) the adjoint of (9), $\bar{\bar{\alpha}}^a$ should also be chosen such that $\bar{\bar{\alpha}}^a=(\bar{\bar{\alpha}}^T)^p$.

Physically, the vectors $\bar{\beta}$, $\bar{\beta}^a$ and $\bar{\gamma}$, $\bar{\gamma}^a$ may be interpreted as electric- and magnetic-surface currents, and the dyadic $\bar{\bar{\alpha}}$ as the permittivity of an infinitesimally thin layer over the surface S. In the expressions for the variational principles given by (19)–(21), (22)–(24), and those for the generalized reactions given by (25), these constants $\bar{\beta}$, $\bar{\beta}^a$, $\bar{\gamma}$, $\bar{\gamma}^a$, and $\bar{\bar{\alpha}}$ have actually served as sheets of additional sources and sheet of extra anisotropic material at the boundary S.

The variational equivalent of (9) and (10) may be written in terms of $(\bar{E},\bar{E}^a)$, $(\bar{H},\bar{H}^a)$, or $(\bar{E},\bar{H},\bar{E}^a,\bar{H}^a)$ and is named E-, H-, or E,H- formulation, respectively.

A. E-Formulation

First consider the variational formulation in terms of $\bar{E}$ and $\bar{E}^a$. To this end, the unknowns $\bar{H}$ and $\bar{H}^a$ in the original and adjoint problems (9) and (10) are considered as functions of $\bar{E}$ and $\bar{E}^a$, respectively, i.e.,

$$\bar{H}(\bar{E})=-\bar{\bar{\mu}}^{-1}\cdot(\nabla\times\bar{E}+\bar{M})/j\omega$$
$$\bar{H}^a(\bar{E}^a)=-\bar{\bar{\mu}}^{a^{-1}}\cdot(\nabla\times\bar{E}^a+\bar{M}^a)/j\omega. \tag{14}$$

Then $\bar{H}$ and $\bar{H}^a$ in (9) and (10) may be eliminated to obtain the wave equations as follows:

$$L\bar{E}\equiv-\nabla\times\left(\bar{\bar{\mu}}^{-1}\cdot\nabla\times\bar{E}\right)+\omega^2\bar{\bar{\epsilon}}\cdot\bar{E}$$
$$=j\omega\bar{J}+\nabla\times\left(\bar{\bar{\mu}}^{-1}\cdot\bar{M}\right), \qquad \text{in } V$$
$$\bar{B}(\bar{E},\bar{H}(\bar{E}))=0, \qquad \text{on } S \tag{15}$$

$$L^a\bar{E}^a\equiv-\nabla\times\left(\bar{\bar{\mu}}^{a^{-1}}\cdot\nabla\times\bar{E}^a\right)+\omega^2\bar{\bar{\epsilon}}^a\cdot\bar{E}^a$$
$$=j\omega\bar{J}^a+\nabla\times\left(\bar{\bar{\mu}}^{a^{-1}}\cdot\bar{M}^a\right), \qquad \text{in } V$$
$$\bar{B}^a(\bar{E}^a,\bar{H}^a(\bar{E}^a))=0, \qquad \text{on } S. \tag{16}$$

Here we have the electric field $\bar{E}$ to be determined from the original problem (15) and the adjoint electric field $\bar{E}^a$ to be determined from the adjoint problem (16).

To absorb the boundary conditions into the variational formulation as well as to simplify the derivation, a symbol $\Delta(S)$ is introduced for converting volume integral into surface integral and vice versa,

$$\int_V F(\bar{r})\Delta(S)\,dV=\int_S F(\bar{r})\,dS$$
$$\Delta(S)^*=\Delta(S). \tag{17}$$

In (17), F is a scalar function of the position $\bar{r}$ so it may be replaced by any component of a vector or a tensor.

To seek for the variational formulation of (15) and (16) with boundary conditions (13), we now start from the equation of the form

$$\langle\delta\bar{E}^a, L\bar{E}-j\omega\bar{J}-\nabla\times\left(\bar{\bar{\mu}}^{-1}\cdot\bar{M}\right)\rangle$$
$$+\langle L^a\bar{E}^a-j\omega\bar{J}^a-\nabla\times\left(\bar{\bar{\mu}}^{a^{-1}}\cdot\bar{M}^a\right),\delta\bar{E}\rangle$$
$$+\langle\delta\bar{E}^a, D\bar{B}(\bar{E},\bar{H}(\bar{E}))\Delta(S)\rangle$$
$$+\langle D^a\bar{B}^a(\bar{E}^a,\bar{H}^a(\bar{E}^a))\Delta(S),\delta\bar{E}\rangle=0 \tag{18}$$

and try to reduce the left-hand side of (18) into the first variation δI of some functional I. Note that we have also included the boundary conditions $\bar{B}=\bar{B}_2=0$ and $\bar{B}^a=\bar{B}_2^a=0$, (13), as natural ones into the derivation. The constants D and D^a are chosen $(D=D^a=j\omega)$ so that the dimension in (18) can be matched and the left-hand side of (18) can be expressed as the desired form δI.

Note that (18) can also be applied to the problems (15) and (16) with the boundary conditions (12), $\bar{B}=\bar{B}_1=0$ and $\bar{B}^a=\bar{B}_1^a=0$, regarded as essential ones. In this case, the last two terms in the left-hand side of (18) are then dropped automatically.

For the problems (15) and (16) with the same boundary conditions (12), $\bar{B}=\bar{B}_1=0$ and $\bar{B}^a=\bar{B}_1^a=0$, regarded as natural ones, one should start from (18) with $\delta\bar{E}$ and $\delta\bar{E}^a$ in the surface integral terms replaced by $\delta(j\bar{H})$ and $\delta(j\bar{H}^a)$, and $D=D^a=-\omega$.

By a straightforward manipulation it can be shown, from (18), that the mathematical equivalent of solving (15) and (16) is the variational problem as follows:

$$\delta I^e(\bar{E},\bar{E}^a)=0 \tag{19}$$

where the functional $I^e(\bar{E},\bar{E}^a)$ takes the form

$$I_{B1}^e(\bar{E},\bar{E}^a)=I_{B1E}^e(\bar{E},\bar{E}^a)$$
$$-\langle j\omega\bar{H}^a(\bar{E}^a),(\hat{n}\times\bar{E}-\bar{\gamma})\Delta(S)\rangle$$
$$-\langle(\hat{n}\times\bar{E}^a-\bar{\gamma}^a)\Delta(S),j\omega\bar{H}(\bar{E})\rangle \tag{20a}$$

$$I_{B1E}^{e}(\bar{E},\bar{E}^a)=-\omega^2\langle j\bar{H}^a(\bar{E}^a),\bar{\bar{\mu}}\cdot j\bar{H}(\bar{E})\rangle+\omega^2\langle\bar{E}^a,\bar{\bar{\epsilon}}\cdot\bar{E}\rangle-\langle\bar{E}^a,j\omega\bar{J}\rangle-\langle j\omega\bar{J}^a,\bar{E}\rangle \tag{20b}$$

for the conditions (12), or the form

$$I_{B2}^{e}(\bar{E},\bar{E}^a)=-\omega^2\langle j\bar{H}^a(\bar{E}^a),\bar{\bar{\mu}}\cdot j\bar{H}(\bar{E})\rangle+\omega^2\langle\bar{E}^a,(\bar{\bar{\epsilon}}+\bar{\bar{\alpha}}\Delta(S))\cdot\bar{E}\rangle-\langle\bar{E}^a,j\omega(\bar{J}+\bar{\beta}\Delta(S))\rangle-\langle j\omega(\bar{J}^a+\bar{\beta}^a\Delta(S)),\bar{E}\rangle \tag{21}$$

for the conditions (13). In (20) and (21), $\bar{H}(\bar{E})$ is actually a function of $\bar{E}$, $\bar{H}^a(\bar{E}^a)$ a function of $\bar{E}^a$, as given by (14), and $\Delta(S)$ is the symbol defined by (17).

Precisely, the variational principle states that the problem of solving $\bar{E}$ and $\bar{E}^a$ simultaneously from (15) and (16) with (12) (or (13)) is completely equivalent to that of determining the stationary functions $\bar{E}$ and $\bar{E}^a$ from the variational problem: $\delta I^e=0$, $I^e=I_{B1}^e$ (or I_{B2}^e), with both $\bar{E}$ and $\bar{E}^a$ unrestricted on the boundary surface S. In this case the conditions (12) (or (13)) are just the natural boundary conditions of the variational problem (19). Alternatively, we may solve the variational problem (19) with the trial functions $\bar{E}$ and $\bar{E}^a$ subject to the same constraints (12) $\bar{B}_1=\bar{B}_1^a=0$. With these constraints, the functional I_{B1}^e then reduces to I_{B1E}^e and the same conditions (12) are now the essential boundary conditions of the variational problem.

B. H-Formulation

The formulas for H-formulation, which may be obtained from those for E-formulation by means of duality transformation: $\bar{E}\to j\bar{H}$, $j\bar{H}\to\bar{E}$, $\bar{\bar{\epsilon}}\to\bar{\bar{\mu}}$, $\bar{\bar{\mu}}\to\bar{\bar{\epsilon}}$, $-j\bar{J}\to\bar{M}$, $\bar{M}\to-j\bar{J}$, etc., (or that given by Harrington [9]) are omitted in this investigation.

C. E, H-Formulation

Instead of treating $\bar{H}$ as a function of $\bar{E}$ and $\bar{H}^a$ as a function of $\bar{E}^a$ as did in Section III-A, we may also regard $\bar{E},\bar{H},\bar{E}^a,\bar{H}^a$ as the unrelated unknown fields to be determined from (9) and (10). In this manner, we then have the equivalent variational formulation of (9) and (10) in terms of $\bar{E},\bar{H},\bar{E}^a,\bar{H}^a$

$$\delta I^{e,h}(\bar{E},\bar{H},\bar{E}^a,\bar{H}^a)=0 \tag{22}$$

where the functional $I^{e,h}(\bar{E},\bar{H},\bar{E}^a,\bar{H}^a)$ is

$$I_{B1}^{e,h}(\bar{E},\bar{H},\bar{E}^a,\bar{H}^a)=I_{B1E}^{e,h}(\bar{E},\bar{H},\bar{E}^a,\bar{H}^a)-\langle j\omega\bar{H}^a,(\hat{n}\times\bar{E}-\bar{\gamma})\Delta(S)\rangle-\langle(\hat{n}\times\bar{E}^a-\bar{\gamma}^a)\Delta(S),j\omega\bar{H}\rangle$$

$$I_{B1E}^{e,h}(\bar{E},\bar{H},\bar{E}^a,\bar{H}^a)=\omega^2\langle j\bar{H}^a,\bar{\bar{\mu}}\cdot j\bar{H}\rangle+\omega^2\langle\bar{E}^a,\bar{\bar{\epsilon}}\cdot\bar{E}\rangle+\langle j\omega\bar{H}^a,\nabla\times\bar{E}+\bar{M}\rangle+\langle\nabla\times\bar{E}^a+\bar{M}^a,j\omega\bar{H}\rangle-\langle\bar{E}^a,j\omega\bar{J}\rangle-\langle j\omega\bar{J}^a,\bar{E}\rangle \tag{23}$$

for the conditions (12), or

$$I_{B2}^{e,h}(\bar{E},\bar{H},\bar{E}^a,\bar{H}^a)=\omega^2\langle j\bar{H}^a,\bar{\bar{\mu}}\cdot j\bar{H}\rangle+\omega^2\langle\bar{E}^a,(\bar{\bar{\epsilon}}+\bar{\bar{\alpha}}\Delta(S))\cdot E\rangle+\langle j\omega\bar{H}^a,\nabla\times\bar{E}+\bar{M}\rangle+\langle\nabla\times\bar{E}^a+\bar{M}^a,j\omega\bar{H}\rangle-\langle\bar{E}^a,j\omega(\bar{J}+\bar{\beta}\Delta(S))\rangle-\langle j\omega(\bar{J}^a+\bar{\beta}^a\Delta(S)),\bar{E}\rangle \tag{24}$$

for the conditions (13).

The variational principle again states that the solutions $(\bar{E},\bar{H})$ of (9) and $(\bar{E}^a,\bar{H}^a)$ of (10) with (12) (or (13)) are also the solutions $(\bar{E},\bar{H},\bar{E}^a,\bar{H}^a)$ of the variational problem: $\delta I^{e,h}=0$, $I^{e,h}=I_{B1}^{e,h}$ (or $I_{B2}^{e,h}$) with the trial functions $(\bar{E},\bar{H},\bar{E}^a,\bar{H}^a)$ completely arbitrary on the boundary S, and vice versa. This is the case of regarding (12) (or (13)) as the natural boundary conditions. For the case of regarding the same conditions (12) as the essential ones, the variational problem should be solved from: $\delta I^{e,h}=0, I^{e,h}=I_{B1E}^{e,h}$ with the trial functions $(\bar{E},\bar{H},\bar{E}^a,\bar{H}^a)$ subject to the constraints (12).

Note that by regarding $\bar{H}$ and $\bar{H}^a$ in (23) and (24) as functions of $\bar{E}$ and $\bar{E}^a$, respectively, as given by (14), one may show that the functionals in (23) and (24) for E,H-formulation can be reduced to those in (20) and (21) for E-formulation, respectively. The reduction of the unknowns, from $(\bar{E},\bar{H},\bar{E}^a,\bar{H}^a)$ to $(\bar{E},\bar{E}^a)$, does not raise the order of the derivatives in the resultant expressions, since (20), (21) and (23), (24) all contain the derivatives of at most order one. In this sense, the formulas of E-formulation are more attractive than those of E,H-formulation.

Note also that with $(j\bar{H})$ as well as $(j\bar{H}^a)$ treated as a single quantity in the derivation, the variational formulations for real- and complex-type inner products can both be expressed in the same form as given by (19)–(21) and (22)–(24).

IV. The Generalized Reaction Concept

We now present physical interpretation of the general variational principle described by (19)–(21) and (22)–(24). For this purpose, we define the "generalized reactions" as

$$\langle f^a,s\rangle=j\{\langle j\bar{H}^a,\bar{M}+\bar{M}_s\Delta(S)\rangle-\langle\bar{E}^a,j[\bar{J}+\bar{\beta}\Delta(S)]\rangle\}$$

$$\langle s^a,f\rangle=j\{\langle\bar{M}^a+\bar{M}_s^a\Delta(S),j\bar{H}\rangle-\langle j[\bar{J}^a+\bar{\beta}^a\Delta(S)],\bar{E}\rangle\}$$

$$\langle f^a,s(f)\rangle=j\{\langle j\bar{H}^a,\bar{M}(\bar{E},\bar{H})+\bar{M}_s(\bar{E},\bar{H})\Delta(S)\rangle-\langle\bar{E}^a,j[\bar{J}(\bar{E},\bar{H})+\bar{\beta}(\bar{E},\bar{H})\Delta(S)]\rangle\}$$

$$\langle s^a(f^a),f\rangle=j\{\langle\bar{M}^a(\bar{E}^a,\bar{H}^a)+\bar{M}_s^a(\bar{E}^a,\bar{H}^a)\Delta(S),j\bar{H}\rangle-\langle j[\bar{J}^a(\bar{E}^a,\bar{H}^a)+\bar{\beta}^a(\bar{E}^a,\bar{H}^a)\Delta(S)],\bar{E}\rangle\}. \tag{25}$$

Then it can be shown that the functionals $I^e(\bar{E},\bar{E}^a)$ in (20), (21) and $I^{e,h}(\bar{E},\bar{H},\bar{E}^a,\bar{H}^a)$ in (23), (24) can be rewritten in terms of these generalized reactions as follows:

$$I(f,f^a)=j\omega[\langle f^a,s(f)\rangle-\langle s^a,f\rangle-\langle f^a,s\rangle]. \tag{26}$$

Note that (26) is actually an extension of (5) if one defines a field dependent source such that $s(f)=Lf$ in (5).

In (25), $(\bar{J},\bar{M})$ and $(\bar{J}^a,\bar{M}^a)$ are the true sources of the original and adjoint problems, (9) and (10), respectively. The fields $(\bar{E},\bar{H})$ and $(\bar{E}^a,\bar{H}^a)$ are the trial fields used in the original and adjoint systems (9) and (10). The field-dependent sources

$$\bar{J}(\bar{E},\bar{H})=\nabla\times\bar{H}-j\omega\bar{\bar{\epsilon}}\cdot\bar{E}$$
$$\bar{M}(\bar{E},\bar{H})=-\nabla\times\bar{E}-j\omega\bar{\bar{\mu}}\cdot\bar{H} \tag{27}$$

are the ones to support the trial fields $(\bar{E},\bar{H})$, and

$$\bar{J}^a(\bar{E}^a,\bar{H}^a)=\nabla\times\bar{H}^a-j\omega\bar{\bar{\epsilon}}^a\cdot\bar{E}^a$$
$$\bar{M}^a(\bar{E}^a,\bar{H}^a)=-\nabla\times\bar{E}^a-j\omega\bar{\bar{\mu}}^a\cdot\bar{H}^a \tag{28}$$

are those to support the trial fields $(\bar{E}^a,\bar{H}^a)$.

The electric surface currents $\bar{\beta}$, $\bar{\beta}^a$, $\bar{\beta}(\bar{E},\bar{H})$, $\bar{\beta}^a(\bar{E}^a,\bar{H}^a)$ and magnetic surface currents $\bar{M}_s$, $\bar{M}_s^a$, $\bar{M}_s(\bar{E},\bar{H})$, $\bar{M}_s^a(\bar{E}^a,\bar{H}^a)$ in (25) are chosen according to the problems. For the problem with the boundary conditions $\bar{B}_1=\bar{B}_1^a=0$, (12), the currents are chosen such that

$$\bar{\beta}=\bar{\beta}^a=\bar{\beta}(\bar{E},\bar{H})=\bar{\beta}^a(\bar{E}^a,\bar{H}^a)=0$$
$$\bar{M}_s(\bar{E},\bar{H})=\hat{n}\times\bar{E}$$
$$\bar{M}_s^a(\bar{E}^a,\bar{H}^a)=\hat{n}\times\bar{E}^a \tag{29}$$

and $\bar{M}_s=\hat{n}\times\bar{E}$, $\bar{M}_s^a=\hat{n}\times\bar{E}^a$ or $\bar{M}_s=\bar{\gamma}$, $\bar{M}_s^a=\bar{\gamma}^a$ depending on whether the conditions (12) are regarded as essential or natural ones. For the problem with the natural boundary conditions $\bar{B}_2=\bar{B}_2^a=0$, (13), the currents should be chosen such that

$$\bar{M}_s=\bar{M}_s^a=\bar{M}_s(\bar{E},\bar{H})=\bar{M}_s^a(\bar{E}^a,\bar{H}^a)=0$$
$$\bar{\beta}(\bar{E},\bar{H})=-j\omega\bar{\bar{\alpha}}\cdot\bar{E}+\bar{H}\times\hat{n}$$
$$\bar{\beta}^a(\bar{E}^a,\bar{H}^a)=-j\omega\bar{\bar{\alpha}}^a\cdot\bar{E}^a+\bar{H}^a\times\hat{n} \tag{30}$$

and $\bar{\beta},\bar{\beta}^a$ are defined by (13).

The generalized reactions in (26) should be calculated from (25), (27)–(30) with $\bar{J},\bar{M},\bar{J}^a,\bar{M}^a$ denoting the true sources and $\bar{E},\bar{H},\bar{E}^a,\bar{H}^a$ denoting the trial fields.

The expressions in (25)–(30) are mainly written for the functionals $I^{e,h}(\bar{E},\bar{H},\bar{E}^a,\bar{H}^a)$ of E,H-formulation. The same expressions may be applied to the functionals $I^e(\bar{E},\bar{E}^a)$ of E-formulation if one regards $\bar{H}$ as a function of $\bar{E}$ and $\bar{H}^a$ as a function of $\bar{E}^a$ as given by (14). Then $\bar{M}(\bar{E},\bar{H})$ and $\bar{M}^a(\bar{E}^a,\bar{H}^a)$ automatically reduce to the true sources $\bar{M}$ and $\bar{M}^a$, respectively. Note that (25) and (26) are invariant under duality transformation, thus, they are also applicable to the expressions for H-formulation.

The term $\langle f^a, s\rangle$ may be interpreted as the "generalized reaction" of $(\bar{E}^a,\bar{H}^a)$, the trial fields in the adjoint system (10), on $(\bar{J}+\bar{\beta}\Delta(S),\ \bar{M}+\bar{M}_s\Delta(S))$, the sources of the original problem (9). The term $\langle f^a, s(f)\rangle$ is the reaction of the trial fields $(\bar{E}^a,\bar{H}^a)$ on the sources $(\bar{J}(\bar{E},\bar{H})+\bar{\beta}(\bar{E},\bar{H})\Delta(S),\ \bar{M}(\bar{E},\bar{H})+\bar{M}_s(\bar{E},\bar{H})\Delta(S))$ to support the trial fields $(\bar{E},\bar{H})$. Similar interpretations may be applied to $\langle s^a, f\rangle$ and $\langle s^a(f^a), f\rangle$.

Note that we have included surface sources (currents) $\bar{\beta}, \bar{\beta}^a,\cdots,\bar{M}_s, \bar{M}_s^a,\cdots$ and surface materials $\bar{\bar{\alpha}}$, $\bar{\bar{\alpha}}^a$ in the definition of the reactions. The generalized reactions (25) reduce to the conventional ones proposed by Rumsey [8] when the real-type inner product is adopted in formulation and the surface quantities $\bar{\bar{\alpha}}$, $\bar{\bar{\alpha}}^a$, $\bar{\beta}$, $\bar{\beta}^a,\cdots,\bar{M}_s$, $\bar{M}_s^a,\cdots$ are removed from the above equations.

The general variational principle described by (22)–(24) (or (19)–(21)) now has a physical interpretation as follows: the true solutions (fields) of the original and adjoint problems (9) and (10) (or (15) and (16)) are just the ones that give the sum of the generalized reactions in (26) a stationary value.

The trial fields $(\bar{E},\bar{H})$ and $(\bar{E}^a,\bar{H}^a)$ used in the original and adjoint systems possess the following symmetric property:

$$\langle f^a, s(f)\rangle=\langle s^a(f^a),f\rangle. \tag{31}$$

In words, the reaction of the trial fields $(\bar{E}^a, \bar{H}^a)$ (in the adjoint system) on the equivalent sources to produce the trial fields $(\bar{E}, \bar{H})$ (in the original system) is always equal to that of $(\bar{E}, \bar{H})$ on the equivalent sources to produce $(\bar{E}^a, \bar{H}^a)$.

Note that for the problems (9) and (10) with the boundary conditions (12) (or (13)), the reaction $\langle f^a, s(f)\rangle_t$ is equal to $\langle f^a, s\rangle_t$ and $\langle s^a(f^a), f\rangle_t$ is equal to $\langle s^a, f\rangle_t$ whenever the trial fields are equal to the true fields. The subscript t in $\langle\ ,\ \rangle_t$ indicates that the true fields and sources of (9) and (10) are utilized in the expressions for the reactions. One consequence is that the true fields and sources of the original and adjoint problems, (9) and (10), should be connected by the relation

$$\langle f^a, s\rangle_t=\langle s^a, f\rangle_t. \tag{32}$$

This is the generalized reciprocity theorem [29] which states that the reaction of the true fields of the adjoint system (10) on the true sources of the original system (9) should be equal to that of the true fields of (9) on the true sources of (10). Another consequence is that the functional I in (26) has a stationary value equal to $-j\omega\langle f^a, s\rangle_t$ $(=-j\omega\langle s^a, f\rangle_t)$.

It remains to consider the important special case of a self-adjoint problem such that

$$(\bar{\bar{\epsilon}}^a, \bar{\bar{\mu}}^a, \bar{\bar{\alpha}}^a)=(\bar{\bar{\epsilon}}, \bar{\bar{\mu}}, \bar{\bar{\alpha}}). \tag{33}$$

The dyadics in (33) are then symmetric or Hermitian depending on whether real- or complex-type inner product is adopted. For this case, there are many choices for the adjoint quantities. A convenient one is that

$$(\bar{J}^a,\bar{M}^a,\bar{E}^a,\bar{H}^a,\bar{\beta}^a,\bar{\gamma}^a)=(\bar{J},\bar{M},\bar{E},\bar{H},\bar{\beta},\bar{\gamma}). \tag{34}$$

Then the functional I in (26) can be written in terms of self-reactions as follows:

$$I(f)=j\omega[\langle f, s(f)\rangle-\langle s,f\rangle-\langle f, s\rangle]. \tag{35}$$

For the self-adjoint problem discussed by the complex-type inner product, the expressions in (20) and (21) then automatically reduce to the ones studied by the previous investigator [10]. In particular, the variational expression I^e_{B1E} in (20) has an interesting physical interpretation when it is rewritten in the following form:

$$\frac{1}{2\omega} I^{e}_{B1E}(\bar{E}) = \omega\left[1/2\langle \bar{E}, \bar{\bar{\epsilon}}\cdot\bar{E}\rangle - 1/2\langle \bar{H}(\bar{E}), \bar{\bar{\mu}}\cdot\bar{H}(\bar{E})\rangle\right] + \left(-\mathrm{Im}\langle \bar{J}, \bar{E}\rangle\right) \quad (36)$$

where Im means the imaginary part of a complex quantity. The first and second terms in the right-hand side of (36) may be interpreted as the time-average electric and magnetic energy stored within the region V, and the third term as the reactive power supplied by the electric source $\bar{J}$. The variational principle for an electromagnetic system then states that the true solution $\bar{E}$ of the variational problem $\delta I^{e}_{B1E}=0$ should be the one that makes the difference in the energy, electric minus magnetic, of the system plus the (electric) source reactive power a stationary value. The same interpretation may be applied to the expression in (21) if the surface quantities $\bar{\bar{\alpha}}$ and $\bar{\beta}$ are properly included in (36) (i.e., $\bar{\bar{\epsilon}} \to \bar{\bar{\epsilon}} + \bar{\bar{\alpha}}\Delta(S)$ and $\bar{J} \to \bar{J} + \bar{\beta}\Delta(S)$).

Using the complex-type inner product to describe a self-adjoint problem it can also be shown that the self-reactions $\langle f, s(f)\rangle$ and $\langle s(f), f\rangle$ are all imaginary, since

$$\langle f, s(f)\rangle = \langle s(f), f\rangle = -\langle f, s(f)\rangle^{*}. \quad (37)$$

This implies that the self-reaction such as $\langle f,s\rangle_t$ has no real part. Physically, the imaginary part of $\langle f,s\rangle_t$ is just the reactive power delivered by the true magnetic source $\bar{M}+\bar{M}_s\Delta(S)$ minus that delivered by the true electric source $\bar{J}+\bar{\beta}\Delta(S)$. These reactive powers are exactly the stationary value of the functional $I/(-j\omega)$.

Note that the expression in (35) is actually an alternative version of that given by (36). The sum of the reactive powers in (35) are, therefore, related to the difference of the average stored energy plus the (electric) source reactive power given by (36).

The electromagnetic variational (or stationary) principle described by $\delta I = 0$ (where I is defined by (20), (21), (23), (24), (26), (35), and (36)) may be considered as an extension of the principle of least action in physics. The principle now states that the true solution of a self-adjoint (or non-self-adjoint) electromagnetic system is just the one that makes the sum of the reactive powers in (35) or (36) (or the sum of the generalized reactions in (26)) a stationary value.

Another choice of the adjoint quantities for a self-adjoint problem is worthy of further investigation. If the self-adjoint problem is discussed by the real-type inner product, then one may use the alternative choice of the form

$$(\bar{J},\bar{M},\bar{E},\bar{H},\bar{\beta},\bar{\gamma}) = (\bar{J}_1,\bar{M}_1,\bar{E}_1,\bar{H}_1,\bar{\beta}_1,\bar{\gamma}_1)$$
$$(\bar{J}^a,\bar{M}^a,\bar{E}^a,\bar{H}^a,\bar{\beta}^a,\bar{\gamma}^a) = (\bar{J}_2,\bar{M}_2,\bar{E}_2,\bar{H}_2,\bar{\beta}_2,\bar{\gamma}_2). \quad (38)$$

The generalized reactions defined by (25) then automatically reduce to the ones originally proposed by Rumsey [8], [9]. The generalized reciprocity theorem (32) also reduces to the conventional one [9]

$$\langle f_2, s_1\rangle_t = \langle f_1, s_2\rangle_t \quad (39)$$

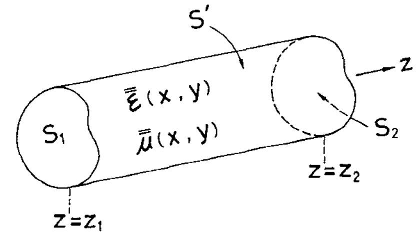

Fig. 2. A section of infinite uniform guide for establishing two-dimensional formulation.

for relating two different problems $(\bar{J}_1,\bar{M}_1,\bar{E}_1,\bar{H}_1)$ and $(\bar{J}_2,\bar{M}_2,\bar{E}_2,\bar{H}_2)$ in the same environment.

V. Reduction to Two-Dimensional Problems

We now apply the three-dimensional variational expressions in Sections III and IV to the derivation of the two-dimensional ones for an infinite uniform guide (Fig. 2). Although these expressions can also be obtained directly from the two-dimensional governing differential equations, it is interesting to establish the same results with a different approach.

Consider a section of uniform guide, from $z=z_1$ to $z=z_2$, to form a three-dimensional region (V of Fig. 1) with the boundary S consisting of the surface S' plus two cross-sectional planes S_1 and S_2 at $z=z_1$ and $z=z_2$, respectively, as shown in Fig. 2. Note that the two-dimensional problem in Fig. 2 should only include S' as its boundary, consequently, the starting equation (18) for this purpose should not contain the surface integral terms over S_1 and S_2.

Let us consider the variational expressions for the cases

$$\left[\bar{E}(x,y,z),\bar{H}(x,y,z),\bar{\beta}(x,y,z),\bar{\gamma}(x,y,z)\right] = \left[\bar{E}(x,y),\bar{H}(x,y),\bar{\beta}(x,y),\bar{\gamma}(x,y)\right]e^{-jkz} \quad (40)$$

$$\left[\bar{E}^a(x,y,z),\bar{H}^a(x,y,z),\bar{\beta}^a(x,y,z),\bar{\gamma}^a(x,y,z)\right] = \left[\bar{E}^a(x,y),\bar{H}^a(x,y),\bar{\beta}^a(x,y),\bar{\gamma}^a(x,y)\right]e^{-jk^a z}. \quad (41)$$

Note that only under the condition

$$(jk^a)^p = -jk \quad (42)$$

it is possible to get a useful expression such as (43). By substituting (40) and (41) into (18) with the condition (42) it is discovered that the left-hand side of (18) (i.e., the volume integral plus the surface integral over S') may be written as

$$(z_2 - z_1)\delta K = 0 \quad (43)$$

where K is a functional $K(\bar{E}(x,y),\ \bar{H}(x,y),\ \bar{E}^a(x,y),\ \bar{H}^a(x,y))$ independent of z.

The two-dimensional functional K is related to the three-dimensional one I defined by (20), (21), (23), (24), (26), (35), or (36) with the fields given by (40)–(42), the boundary conditions on S' specified by (12) or (13), and some suitable boundary conditions, for example (12) or (13), specified on S_1 and S_2. The relation is

$$K = dI/dz_2 \quad (44)$$

i.e., the two-dimensional functional K can be obtained from the three-dimensional one I by performing the integration with respect to x, y, and z to get I first and then taking the derivative of I with respect to z_2. Actually, this functional K can directly be obtained from I by simply removing the integration with respect to z.

In conclusion, we have the two-dimensional formulation for a guide

$$\delta K = 0 \tag{45}$$

since $(z_2 - z_1)$ in (43) is arbitrary.

By using the complex-type inner product to discuss a self-adjoint problem and choosing the adjoint quantities according to (34), it is found that the condition (42) is fulfilled automatically. Then one can yield, from (44), the same results investigated by the previous authors [7], [11]–[15].

VI. Applications of the Variational Principle

Examples of deriving the variational expressions for fields computation and establishing the stationary formulas for certain physical quantities, such as the eigenvalues, will be included to demonstrate the applications of the variational principle.

Consider first the normal incidence propagation problem for an isotropic inhomogeneous dielectric slab [16]. From the region 1 ($x<0$) of homogeneous medium $(\mu_0, \epsilon_0\epsilon_1)$, a uniform plane wave is normally incident upon the region 2 ($0<x<a$) of inhomogeneous medium $(\mu_0, \epsilon_0\epsilon_2(x))$, and is then transmitted to the region 3 ($x>a$) of homogeneous medium $(\mu_0, \epsilon_0\epsilon_3)$. By means of the continuity conditions for tangential electric and magnetic fields over the boundaries at $x=0$ and $x=a$, one can obtain the constants associated with the problem as follows:

$$\bar{\bar{\alpha}}(0) = \frac{k_1}{j\omega^2\mu_0}\hat{y}\hat{y} \qquad \bar{\beta}(0) = \frac{-2k_1}{\omega\mu_0}\hat{y}$$
$$\bar{\bar{\alpha}}(a) = \frac{k_3}{j\omega^2\mu_0}\hat{y}\hat{y} \qquad \bar{\beta}(a) = 0. \tag{46}$$

For computational simplicity by adopting the real-type inner product and choosing (34) one has, from (21), the variational expression for computing the electric field $\bar{E}(=\hat{y}\psi(x))$

$$-\mu_0 I(\psi) = \int_0^a \left[\psi'(x)^2 - k_0^2\epsilon_2(x)\psi^2(x)\right]dx + jk_3\psi^2(a) + jk_1\psi^2(0) - j4k_1\psi(0). \tag{47}$$

The constants ϵ_0 and μ_0 are the permittivity and permeability of free space; ϵ_1, $\epsilon_2(x)$, and ϵ_3 are the relative permittivities of region 1, 2, and 3; and k_0, k_1, and k_3 are the propagation constants of free space, region 1 and region 3, respectively.

For the functional $I(f, f^a)$ in (26) (or (5)) which contains an unknown physical parameter such as resonance frequency, cutoff frequency, or propagation constant, the stationary formula for that parameter may be obtained from the equation

$$I(f, f^a) = I(f_t, f_t^a) \tag{48}$$

where f, f^a are the trial fields and f_t, f_t^a are the true fields of the original and adjoint problems. Note that the functional (26) for the true fields is zero, i.e.,

$$I(f_t, f_t^a) = 0 \tag{49}$$

whenever the true sources $\bar{J}$, $\bar{M}$, $\bar{J}^a$, $\bar{M}^a$ and the surface sources $\bar{\beta}$, $\bar{\beta}^a$, $\bar{M}_s$, $\bar{M}_s^a$ are all zero.

It remains to apply (48) and (49) to the problems of establishing the stationary formulas for a cavity resonator and a waveguide.

Consider the resonator which has a perfectly conducting wall S to enclose a region V of Hermitian medium. The complex type inner product is adopted in formulation. Then the problem is self-adjoint. The adjoint quantities are chosen according to (34). Note that the true sources $(\bar{J}, \bar{M})$ are zero within V so that the relation in (49) can be satisfied. By considering the boundary condition, $\hat{n}\times\bar{E}(S) = \bar{\gamma} = 0$, as a natural one and using (14), (20), (34), (48), and (49), one then has the stationary formula for resonance frequency ω [15] of the form

$$\omega^2 = \left\{ \int_V \nabla\times\bar{E}^*\cdot\bar{\bar{\mu}}^{-1}\cdot\nabla\times\bar{E}\,dV - \int_S \hat{n}\cdot\left[\bar{E}\times\left(\bar{\bar{\mu}}^{-1}\cdot\nabla\times\bar{E}\right)^*\right]dS - \int_S \hat{n}\cdot\left[\bar{E}^*\times\left(\bar{\bar{\mu}}^{-1}\cdot\nabla\times\bar{E}\right)\right]dS \right\} \Big/ \int_V \bar{E}^*\cdot\bar{\bar{\epsilon}}\cdot\bar{E}\,dV. \tag{50}$$

Finally consider the waveguide with a perfectly conducting wall S' to enclose the anisotropic medium

$$\begin{aligned}\bar{\bar{\epsilon}} = \bar{\bar{\epsilon}}(x,y) &= \epsilon_{xx}\hat{x}\hat{x} + \epsilon_{xy}\hat{x}\hat{y} + \epsilon_{xz}\hat{x}\hat{z} \\ &+ \epsilon_{yx}\hat{y}\hat{x} + \epsilon_{yy}\hat{y}\hat{y} + \epsilon_{yz}\hat{y}\hat{z} \\ &- \epsilon_{xz}\hat{z}\hat{x} - \epsilon_{yz}\hat{z}\hat{y} + \epsilon_{zz}\hat{z}\hat{z} \\ \bar{\bar{\mu}} = \bar{\bar{\mu}}(x,y) &= \mu_{xx}\hat{x}\hat{x} + \mu_{xy}\hat{x}\hat{y} + \mu_{xz}\hat{x}\hat{z} \\ &+ \mu_{yx}\hat{y}\hat{x} + \mu_{yy}\hat{y}\hat{y} + \mu_{yz}\hat{y}\hat{z} \\ &- \mu_{xz}\hat{z}\hat{x} - \mu_{yz}\hat{z}\hat{y} + \mu_{zz}\hat{z}\hat{z}.\end{aligned} \tag{51}$$

With the fields expressed as

$$\bar{E}(x,y,z) = \bar{E}(x,y)e^{-jkz} = \left(\bar{E}_t + \hat{z}E_z\right)e^{-jkz}$$
$$\bar{H}(x,y,z) = \bar{H}(x,y)e^{-jkz} = \left(\bar{H}_t + \hat{z}H_z\right)e^{-jkz} \tag{52}$$

and adopting real-type inner product it can be shown, from Maxwell's equations and (42), that the adjoint fields may be written as

$$\bar{E}^a(x,y,z) = \bar{E}^a(x,y)e^{jkz} = \left(\bar{E}_t - \hat{z}E_z\right)e^{jkz}$$
$$\bar{H}^a(x,y,z) = \bar{H}^a(x,y)e^{jkz} = \left(-\bar{H}_t + \hat{z}H_z\right)e^{jkz}. \tag{53}$$

For this specific problem, the relation in (49) is again satisfied. By considering the boundary condition, $\hat{n}\times\bar{E} = \bar{\gamma} = 0$, as an essential one and using (23), (44), (45), (48), and (49) one then has the stationary formula for the propagation constant k:

$$k=\left\{\int_{S'}\left[\omega\bar{E}^a(x,y)\cdot\bar{\bar{\epsilon}}\cdot\bar{E}(x,y)-\omega\bar{H}^a(x,y)\cdot\bar{\bar{\mu}}\cdot\bar{H}(x,y)\right.\right.$$
$$\left.\left.+j\bar{H}^a(x,y)\cdot\nabla_t\times\bar{E}(x,y)+j\bar{H}(x,y)\cdot\nabla_t\times\bar{E}^a(x,y)\right]dS\right\}$$
$$\Big/2\int_{S'}\bar{E}_t\times\bar{H}_t\cdot\hat{z}\,dS. \tag{54}$$

The symbol ∇_t in (54) represents the "transverse" part of the del operator ∇, i.e., $\nabla_t=\nabla-\hat{z}\,\partial/\partial z$. Note that (54) reduces to the special one in [9] whenever $\bar{\bar{\epsilon}}$ and $\bar{\bar{\mu}}$ are scalar.

Stationary formulas for other quantities, such as the impedance of an antenna and the echo area of a scatterer, etc., [7], [9], can also be derived from this variational principle.

VII. Conclusions

The general variational (or stationary) principle has been established for dealing with the non-self-adjoint problem in an electromagnetic system with various boundary conditions. This principle has been interpreted physically in terms of generalized reactions, time-average stored energy, and reactive powers. It has been shown that the true field of a non-self-adjoint system (or a self-adjoint system) is the one that makes the sum of the generalized reactions (or the sum of the reactive powers) a stationary value. In this sense, this general variational principle is indeed an extension of the least action principle in physics. The non-self-adjoint variational expressions have been found automatically leading to the self-adjoint ones discussed by the previous investigators. The general formulation is proved very useful in deriving the variational expressions for fields and eigenvalues computation.

Acknowledgment

The authors are indebted to Prof. C. M. Huang, Y. W. Kiang, and C. T. Liu for helpful discussions, and Miss S. L. Chang for various assistance during the preparation of this manuscript. Valuable suggestions and reference materials from one of the reviewers are much appreciated.

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